Q FACTORS OF FREE AND SHIELDED CYLINDRICAL DIELECTRIC RESONATORS.*

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ABSTRACT

In this paper, the radiation quality factor of cylindrical dielectric resonators has been evaluated by solving the characteristic equation of the chosen mode in the complex plane. In all cases this Q factor is too low for practical applications. So in order to prevent the losses by radiation we enclose the dielectric resonator in a metallic cylindrical shield. For such a geometry the unloaded quality factor of the shielded resonator has been calculated.

INTRODUCTION

In this paper, we present a study of the radiation quality factor of free cylindrical dielectric resonators. The resonators are isolated in free space and are supposed to be free of dielectric losses. The electromagnetic resonance and the radiation quality factor of a sphere and of tubular resonators have been studied by some investigators {1} , {2} , {3} .

Our approach is based as those of Yee {1} and Gastine {2} on the resolution of the characteristic equation of dipolar mode in the complex plane.

From these results, it appears that the use of dielectric resonators at microwave frequencies requires a shield to prevent loss of energy by radiation {6} . For the case of a dielectric cylindrical resonators we have shown that the shield disturbs the resonant frequency and the unloaded quality factor of the dielectric resonator, except in the zone where the diameter of the shield is two or three times that of the dielectric resonator.

The radiation quality factor of the $TE_{0,1,p}$ cylindrical resonator

. Definition

To calculate the radiation quality factor Q_r we introduce the formalism of a complex radianfrequency

$$\omega = \omega' + j\,\omega'' \qquad (1)$$

where : ω' : real pulsation of the mode considered, and ω'' an imaginary part which corresponds to losses. ω'' represents a damping constant.

In order to compute the radiation quality factor, the dielectric is supposed to be lossless. The damping must be attributed entirely to the loss of energy by radiation. We may define the corresponding Q value for this loss as :

$$Q_r = \frac{\omega'}{2\omega''} \qquad (2)$$

and this is called the radiation Q of the resonator.

To solve the problem of the radiation of a cylindrical resonator, it may be necessary to consider two cases which depend on the value of the ratio D/H.

For low D/H, most of the radiation is transversal so we introduce the complex pulsation (1) into the eigenvalue equation derived from the 2_a approximation (only the lateral surface is a perfect magnetic wall) to obtain the Q factor due to the transverse radiation Q_{rt}.

For large D/H, most of the radiation is longitudinal so we introduce the complex pulsation (1) into the eigenvalue equation derived from the 2_H approximation (only the flat surfaces are perfect magnetic walls) to obtain the Q factor due to the longitudinal radiation Q_{rl}.

The next step of the calculation will be to evaluate the radiation Q factor by taking into account that all the surfaces of the dielectric resonator are imperfect magnetic walls. So, in order to approximate this, we suppose that the resultant radiation Q(Q_r) is given by the following relation :

$$\frac{1}{Q_r} = \frac{1}{Q_{rt}} + \frac{1}{Q_{rl}} \qquad (3)$$

. The characteristic equation for Q_{rl} and Q_{rt}

Consider an homogeneous, cylindrical dielectric resonator (height H, radius a, relative permittivity ε_2).

Q_{rt} : *characteristic equation*

If we assume that only the end flat surfaces satisfy the open circuit boundary conditions, the field components are function of Bessel functions of the first kind (J_n)and outside they depend on Hankel functions of the second kind ($H_n^{(2)}$). The continuity of the tangential components of E.M. field at the separation surface (r=a) leads to a characteristic equation which is complex :

$$k_i a \frac{J_o(k_i a)}{J_1(k_i a)} = - k_o a \frac{H_o^{(2)}(k_o a)}{H_o^{(2)}(k_o a)} \qquad (4)$$

In this relation :

$$k_o^2 = -\left(\frac{\omega}{c}\right)^2 + \beta^2 \qquad (5a)$$

$$k_i^2 = \left(\frac{\omega}{c}\right)^2 \varepsilon_2 - \beta^2 \qquad (5b) \qquad (5)$$

Here $= \frac{p\pi}{H_e}$ where H_e effective height of the resonator {4}

$$C = \frac{1}{\sqrt{\varepsilon_o \mu_o}}$$ light velocity in vacum.

Q_{rl} : *characteristic equation*

If we assume that only the circular surface satisfies the open circuit boundary conditions (figure 1b), we obtain by applying the continuity solutions on the separation surface the characteristic equation {4}

$$\tan\frac{\beta H}{2} = \frac{\alpha}{\beta} \qquad (6)$$

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with $\beta^2 = (\frac{\omega}{c})^2 \varepsilon_2 - k_c^2$

$$\alpha^2 = k_c^2 - (\frac{\omega}{c})^2 \quad (7)$$

Here $k_c = \frac{x_{01}}{a_e}$ where a_e : effective radius of the resonator

x_{01} first root of the Bessel function $J_0(x)=0$ (8)

. Lateral radiation Q (Q_{rt}) : resolution of the characteristic equations (4)

Radiation Q of the TE_{01p} *mode*

We assume that ω is a complex value, so the radial wave propagation may be written from (5a) and (5b)

$$z_1^2 = (k_o a)^2 = -\frac{(X-jY)^2}{\varepsilon_2} + \beta^2 a^2 \quad (9)$$

$$z_2^2 = (k_i a)^2 = (X - jY)^2 - \beta^2 a^2 \quad (10)$$

where $X - jY = \frac{a\sqrt{\varepsilon_2}}{C}(\omega' + j\omega'')$ (11)

and $\beta = \frac{p\pi}{H_e}$

Let us study some special cases obtained for a given value of the relative permittivity ε_2.

. When $k_o a \to 0$ From (4) we have $J_o(k_i a) \to 0$ and $k_i a \to x_{01}$. The roots are real, so no radiation exists. We find again the "confined" modes of the cylindrical resonator of whose surfaces are perfect magnetic walls.

. When $0 < k_o a < 1$. Using the approximations for small arguments of Hankel functions :

$$H_o(z) = 1 - \frac{2}{\pi} j \log \frac{2}{z} \quad (12)$$

$$H_1(z) = z/2 - j\frac{2}{\pi z} \quad (13)$$

and the development of Bessel functions given by :

$$J_n(z) = \sqrt{\frac{2}{\pi z}}\{P(n,z)\cos\chi - Q(n,z)\sin\chi\} \quad (14)$$

with $\chi = z - (\frac{n}{2} + \frac{1}{4})\pi$

$$P(n,z) = \sum_{k=0}^{\infty} (-)^k \frac{(4n^2, 2k)}{(2z)^{2k}}$$

$$Q(n,z) = \sum_{k=0}^{\infty} (-)^k \frac{(4n^2, 2k+1)}{(2z)^{2k+1}} \quad (15)$$

{4} takes the following form (16)

$$Az_2(\pi z_1^2 - 4j) + B z_1^2(\pi - 2j \text{ Log } \frac{2}{\gamma z_1}) = 0 \quad (16)$$

$$A = P(0,z_2)(\cos z_2 + \sin z_2) - Q(0,z_2)(\sin z_2 - \cos z_2)$$

$$B = P(1,z_2)(\sin z_2 - \cos z_2) + Q(1,z_2)(\sin z_2 + \cos z_2)$$

In this expression z_1 and z_2 are respectively given by the relations (9) and (10).

When $k_o a >> 1$ from (9) we have $(\beta a)^2 >> \frac{(X-jY)^2}{\varepsilon_2}$ and we can approximate $k_o a$ by the value (17)

$$z_1 = -\frac{(X-jY)^2}{2\varepsilon_2 \beta a} + \beta a \quad (17)$$

Substituting into (4) the approximations for large arguments of Hankel functions :

$$H_n(z) = e^{-j\chi}\sqrt{\frac{2}{\pi z}}\left(1 + j\frac{4n^2-1}{8z}\right) \quad (18)$$

and taking into account (14), (15) we shall have to solve :

$$Az_2 - B\left\{\frac{(X-jY)^2}{2\varepsilon_2 \beta a} - \beta a\right\} = 0 \quad (19)$$

For any $k_o a$ and in particular for $k_o a \neq \beta a$ which is the practical case encountered in the applications of cylindrical resonator to filter, we have to use the following development of Hankel function :

$$H_n(z) = \sqrt{\frac{2}{\pi z}}\{P(n.z) - jQ(n,z)\} e^{-j\chi} \quad (20)$$

with χ, $P(n,z)$, $Q(n,z)$ given by (14), (15)

Substituing (14) and (20) into (4) we have to solve :

$$z_2 A\{P(1,z_1) - jQ(1,z_1)\} + jz_1\{P(0,z_1) - jQ(0,z_1)\} = 0 \quad (21)$$

For a given value of ε_2, and according to the value of $k_o a$, we solve one of the equations (16), (19), (21) by means of a computer. The values of X and Y obtained are substituted into (2) to obtain the variations of the transverse radiation quality factor Q_{rt} as a function of the ratio D/H which have been drawn on curve 1 (ε_2=100) and on curve 2 (ε_2=38).

Special case : Q_r *of the* TE_{010} *mode*

The pulsation ω being a complex value we have since $\beta = 0$:

$$z_1 = k_o a = \frac{X - jY}{\sqrt{\varepsilon_2}} = \frac{a}{c}(\omega' + j\omega'') \quad (22)$$

$$z_2 = k_i a = X - jY = \frac{a}{c}\sqrt{\varepsilon_2}(\omega' + j\omega'') \quad (23)$$

X and Y are solutions of the equation (4). In order to solve this one we use the approximations for large arguments of Hankel functions (18) and those of Bessel functions given by the following formula (24) :

$$J_n(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{n\pi}{4} - \frac{\pi}{2}\right) \quad (24)$$

For large ε_2 : there are two cases : when $\varepsilon_2 \to \infty$ the product $k_i a$ is infinite and when $\varepsilon_2 \to \infty$ the product $k_i a$ is finite.

z_2 remains finite, $\frac{H_o(z_1)}{H_1(z_1)} \to 0$, the solutions of (4) are real. The separation surface (r=a) between dielectric and air is a perfect magnetic wall. We have resonant confined modes {2} {3} .

z_2 is infinite (when $\varepsilon_2 \to \infty$) $H_1^{(2)}(z_1) \to 0$, the solutions of (4) are complex. We have now non confined modes {2} {3} the Q factors of these are very low.

For any ε_2

To solve the problem in this case, we substitute into (4), $J_o(z_2)$ and $J_1(z_2)$ by their values (14) and (15) and $H_o(z_1)$ and $H_1(z_1)$ by their approximations for small arguments (12) and (13). So we obtain a new complex relation

$$\left(\frac{\pi z_2}{\sqrt{\varepsilon_2}} - 4j\right).(R_N + jI_n)\varepsilon_2 - z_2(R_D + jL_D) \cdots\cdots$$

$$\left\{ -\pi + 2j\left(\mathrm{Log}\,\frac{2\sqrt{\varepsilon_2}}{\gamma} - \log X + j\,\frac{Y}{X}\right)\right\} = 0 \qquad (25)$$

with : $R_N = \mathrm{Ch}\,Y\,\{\sin X\{P(0,z_2) - Q(0,z_2)\} + \cdots\cdots$
$\cos X\,\{P(0,z_2) + Q(0,z_2)\}\}$

$R_D = \mathrm{Ch}\,Y\,\{\sin X\,\{P(1,z_2) + Q(1,z_2)\} - \cdots\cdots$
$\cos X\,\{P(1,z_2) - Q(1,z_2)\}\}$

$I_N = \mathrm{Sh}\,Y\,\{\sin X\,\{P(0,z_2) - Q(0,z_2)\} - \cdots\cdots$ (26)
$\cos X\,\{P(0,z_2) + Q(0,z_2)\}\}$

$I_D = -\mathrm{Sh}\,Y\,\{\sin X\,\{P(1,z_2) - Q(1,z_2) + \cdots\cdots$
$\cos X\,\{P(1,z_2) + Q(1,z_2)\}\}$

The resolution of (25) and (26) allows the determination of X and Y and so the radiation quality factor. On curve n°3 we have drawn the variations of Q_r as a function of the relative permittivity of the sample ε_2.

. Longitudinal radiation Q_{r1} : resolution of the characteristic equation (6)

The complex value of the pulsation (1) is introduced into (6) :

$$u^2 = (\beta H)^2 = (X - jY)^2 - k_c^2 H^2 = u_r + j\,u_i$$

$$v^2 = (\alpha H)^2 = k_c^2 H^2 - \frac{(X - jY)^2}{\varepsilon_2} = v_r + jv_i \qquad (27)$$

$$\text{with } k_c = \frac{x_{o1}}{a_e}$$

The equation (6) can be split into two equations :

$$\frac{\tan u_r - \tan u_r \tanh^2 u_i}{1 + \tan^2 u_r \tanh^2 u_i} = \frac{v_i u_r + u_i v_i}{u_r^2 + u_i^2} \qquad (28)$$

$$\frac{\tan^2 u_r \tanh u_i + \tanh u_i}{1 + \tan^2 u_r \tan^2 u_i} = \frac{v_i u_r - u_i v_i}{u_r^2 + u_i^2} \qquad (29)$$

These equations have been solved, with the values of X and Y obtained, we have calculated the radiation Q factor (Q_{r1}) which varies with the value of the ratio D/H. Curves n°1 and n° 2.

. Conclusion - Experimental results

The resultant radiation quality factor is obtained by substituting the value of Q_{rt} and Q_{r1} into (3). The results (applied to the cases of $\varepsilon_2 = 38$ and $\varepsilon_2 = 100$) of this approximation are given on curves n°1 and n°2.

Some experimental results have been done, which agree well with theoretical ones. We have listed these results in the following table :

ε_2	D_{mm}	H_{mm}	experimental Q_r	theoretical Q_r
100	10	5	80	76
38	10	5	30	28
100	30	10	75	73

The unloaded quality factor of cylindrical resonator placed in a cylindrical metallic cavity

Considering the previous results concerning the radiation quality factor of dielectric cylindrical resonators and those of Pellegrin {5} it appears that for practical applications, it is necessary to enclose the dielectric resonator into a shield to prevent loss of energy by radiation.

This shield must be placed sufficiently far from the resonator, in order that the induced losses in the metallic wall will not be too large.

The dielectric resonator (a_e H_e) is supposed to be put in the center of the cylindrical closed shield (ϕ=b height = L) as shown in figure n°2.

The total stored energy $\overline{W}_t$ is made of two parts : the energy stored inside the resonator $\overline{W}_i$ and the energy stored outside $\overline{W}_o$. The latter consists only of the energy in the standing wave between the resonator and the walls of the cavity.

$$\overline{W}_t = \overline{W}_i + \overline{W}_o \qquad (30)$$

From these considerations, we can give the following expression for the unloaded quality factor :

$$\frac{1}{Q_o} = \frac{1}{\overline{W}_t}\left(\overline{W}_i \,.\, \tan\delta_d + \frac{1}{Q_m}\overline{W}_o\right) \qquad (31)$$

In this relation $Q_d = (\tan\delta_d)^{-1}$ is the Q factor which takes into account the losses of the dielectric material and Q_m the Q factor which takes into account the mean power loss $\overline{P}_m$ in the metallic walls

$$Q_m = \omega\,(\overline{W}_o).(\overline{P}_m)^{-1} \qquad (32)$$

We calculate the fields induced by the dielectric resonator into the walls of the shield ; this one being a metallic cavity. For that the dielectric resonator is represented by a small conducting loop. The equivalent magnetic moment m is supposed to be distributed over all the volume τ of the resonator {5}

The energy quantities may be computed from the following relation :

$$\overline{W}_u = \frac{\varepsilon_o}{2}\int E_u E_u^* \, d\tau_u \qquad (33)$$

with u = 1 when we calculate the energy stored outside the sample ($\overline{W}_o$) and u = 2 when we calculate the energy stored inside the sample ($\overline{W}_i$).

$$\overline{W}_i = \frac{2\omega^2\mu_o}{c^2 . x_{o1}^2}\,\pi a_e^4 H_o^2 \{J_1'^2(x_{o1}) + \cdots\cdots$$
$$\left(1 - \frac{1}{x_{o1}^2}\right) J_1^2(x_{o1})\}\left\{\frac{\varepsilon_2 H_e}{2}\left(1 + \frac{\sin\beta H_e}{\beta H_e}\right)\right\} \qquad (34)$$

The dielectric disk resonator has its axis along the center line of the cylindrical cavity. Because of the cylindrical symmetry, only circular electric modes designated $TE_{0,m}$ are excited by the dielectric resonator.

The problem of excitation of a structure by a current distribution has been studied {5} . It is possible to evaluate the electric and magnetic fields components outside the sample. They are expressed as a superposition of the fields $E_{m,n}$ and $H_{m,n}$ of the empty waveguide.

$$E^+ = \sum a_{mn} E_{mn}^+ \qquad E^- = \sum b_{mn} E_{mn}^- \qquad (35)$$

$$H^+ = \sum a_{mn} H_{mn}^+ \qquad H^- = \sum b_{mn} H_{mn}^- \qquad (36)$$

with :

$$a_{mn} = \frac{j\omega\mu_o}{2}\int m\; H_{mn} \,.\, d\tau \qquad (37)$$

$$b_{mn} = \frac{j\omega\mu_o}{2}\int m\; H_{mn} \,.\, d\tau \qquad (38)$$

Knowing the field components we are able to evaluate the energy $\overline{W_o}$ and the power loss $\overline{P_m}$ in the metallic walls of the cylindrical cavity.

$$\overline{W_o} = \frac{0.405}{18 x_{11}^4}\, \varepsilon_o\, \mu_o^4\, \omega^4\, \frac{b^4}{4}\, H_o^4 \{\int m d\tau\}^2\, K_1 \qquad (39)$$

$$\overline{P_m} = 8\sqrt{\frac{\omega\mu}{2\sigma_m}}\, \omega^2\, \mu_o^2\, \pi H_o^4 \{\frac{\pi^4}{x_{11}^4}\, \frac{b^4}{L^4} \cos^2\beta\, \frac{L}{2}\, K_1 \qquad (40)$$

$$+ \frac{bL}{4}\, J_o^4(x_{11})(1 + \frac{\sin \beta L}{\beta L})\} \{\int m\, d\tau\}^2$$

σ_m : metal conductivity.

$$K_1 = 3 \log b - 4(\cos 4.7\, ci\,(2x_{11}) - \sin 4.7\, si(2x_{11})) - \cos 9.4\, ci(4x_{11}) - \sin 9.4\, si\,(4.x_{11})$$

where ci(x) and si(x) are respectively integral cosinus and sinus functions.

Substituting (39) and (40) into (32) we determine Q_m. This value is reported both those of $\overline{W_i}$ and $\tan \delta_d$ into (31) in order to find an expression for the unloaded quality factor Q_o.

With the aid of a computer we have solved this expression, which gives the variations of the Q factor as a function of the parameters of the shield. Curve n°4

On this curve, we can note that :

. if $\frac{b}{a}=1$, the Q factor obtained is the Q_o of the metallic cavity entirely loaded by the dielectric sample. The resonant frequency is that of the loaded cavity.

. if $1.2< \frac{b}{a} <2.6$ The Q factor does not vary rapidly and it is equal about to Q_d. The resonant frequency is that of the dielectric resonator.

. if $\frac{b}{a} > 7$ the Q factor is the unloaded quality factor of the empty metallic cavity. The resonant frequency is that of the empty cavity.

CONCLUSION

In this paper, we have studied the variations of the radiation quality factor of a cylindrical resonator as a function of D/H. From these results, it appears that when the radiation quality factor is always very low, the radiation is important, so it is not possible to use a dielectric resonator in microwave devices without shield. Radiation losses can be suppressed by mean of a closed shield. When the dimensions of this shield are correctly chosen, the losses will be essentially those of the dielectric resonator itself. Then two applications can be envisaged :

- accurate determination of microwave permittivity of dielectric materials
- realization of low insertion losses microwave integrated filters.

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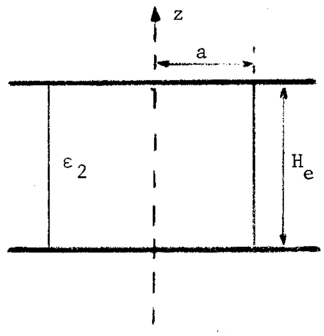

Figure 1a

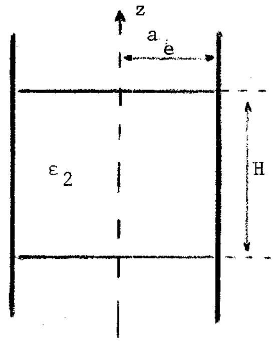

Figure 1b

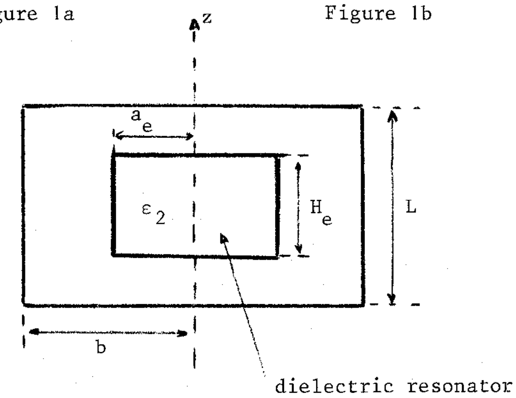

Figure 2